

# RINGS WHOSE INDECOMPOSABLE MODULES ARE PURE-PROJECTIVE OR PURE-INJECTIVE

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**ABSTRACT.** It is proven each ring  $R$  for which every indecomposable right module is pure-projective is right pure-semisimple. Each commutative ring  $R$  for which every indecomposable module is pure-injective is a clean ring and for each maximal ideal  $P$ ,  $R_P$  is a maximal valuation ring. Complete discrete valuation domain of rank one are examples of non-artinian semi-perfect rings with pure-injective indecomposable modules.

In this short note we give a partial answer to the following question posed by D. Simson in [5, Problem 3.2]:

”Give a characterization of rings  $R$  for which every indecomposable right  $R$ -module is pure-projective or pure-injective. Is every such a semi-perfect ring  $R$  right artinian or right pure-semisimple?”

From a result of Stenström we easily deduce that each ring  $R$  whose indecomposable right modules are pure-projective is right pure-semisimple.

**Theorem 1.** *Let  $R$  be a ring for which each indecomposable right module is pure-projective. Then  $R$  is right pure-semisimple.*

*Proof.* Let  $M$  be a non-zero right  $R$ -module. By [1, Proposition 1.13]  $M$  is pure-projective if each pure submodule  $N$  for which  $M/N$  is indecomposable is a direct summand. This last property holds since each indecomposable right  $R$ -module is pure-projective. Hence  $M$  is pure-projective and  $R$  is right pure-semisimple.  $\square$

Now we give an example of a non-artinian commutative semi-perfect ring whose indecomposable modules are pure-injective.

**Example 2.** Let  $R$  be a complete discrete valuation domain of rank one and let  $Q$  be its quotient field. By [4, Corollary 2 p.52] each indecomposable module is cyclic or isomorphic to a factor of  $Q$  and these modules are pure-injective. Clearly  $R$  is not artinian.

**Theorem 3.** *Let  $R$  be a commutative ring for which each indecomposable module is pure-injective. Then  $R$  is a clean ring and  $R_P$  is a maximal valuation ring for each maximal ideal  $P$ . Moreover, each indecomposable  $R$ -module is uniserial.*

*Proof.* By [6, Theorem 9(3)] each indecomposable pure-injective module has a local endomorphism ring. So, if each indecomposable module is pure-injective, then, by [2, Theorem III.7],  $R$  is a clean ring and  $R_P$  is a valuation ring for each maximal

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ideal  $P$ . Since  $R_P$  is indecomposable for each maximal ideal  $P$ , it is pure-injective. It follows that  $R_P$  is maximal for each maximal ideal  $P$ .

Let  $M$  be an indecomposable  $R$ -module. By [2, Proposition III.1], there exists a unique maximal ideal  $P$  such that  $M = M_P$  and its structures of  $R$ -module and  $R_P$ -module coincide. So, we may assume that  $R$  is local. By [3, Theorem 5.4]  $M$  is either the pure-injective hull of an uniserial module or an essential extension of a cyclic module. Since  $R$  is maximal, it follows that  $M$  uniserial.  $\square$

**Remark 4.** It remains to characterize maximal valuation rings for which every indecomposable module is uniserial.

**Example 5.** If  $R$  is the ring defined in [2, Example III.8] or if  $R$  satisfies the equivalent conditions of [2, Theorem III.4] then  $R$  is a ring for which each indecomposable module is pure-injective and uniserial.

**Corollary 6.** *Let  $R$  be a perfect commutative ring for which each indecomposable module is pure-injective. Then  $R$  is pure-semisimple.*

*Proof.*  $R$  is a finite product of artinian valuation rings by Theorem 3. Hence  $R$  is pure-semisimple.  $\square$

Now we give an example of ring  $R$  whose indecomposable right modules are pure-injective and for which there exists an indecomposable left module which is not pure-injective.

**Example 7.** Let  $K$  be a field, let  $V$  be a vector space over  $K$  which is not of finite dimension, let  $S = \text{End}_K(V)$ , let  $J$  be the set of finite rank elements of  $S$  and let  $R$  be the  $K$ -subalgebra of  $S$  generated by  $J$ . Then,  $R$  is Von Neumann regular,  $V$  and  ${}_R R/J \cong K$  are the sole types of simple left modules, and  $V^* = \text{Hom}_K(V, K)$  and  $R_R/J \cong K$  are the sole types of simple right modules. It is easy to check that  $V^*$  and  $R_R/J$  are injective, whence  $R$  is a right V-ring. On the other hand each right  $R$ -module contains a simple module. So, every indecomposable right module is simple. It follows that every indecomposable right module is pure-injective (injective). It is well known that  $V$  is FP-injective,  $V^{**}$  is the injective hull of  $V$  and  $V \subset V^{**}$ . So,  $V$  is not pure-injective.

## REFERENCES

- [1] G. Azumaya. Countabled generatedness version of rings of pure global dimension 0. volume 168 of *Lecture Notes Series*, pages 43–79. Cambridge University Press, (1992).
- [2] F. Couchot. Indecomposable modules and Gelfand rings. *Comm. Algebra*, 35(1):231–241, (2007).
- [3] A. Facchini. Relative injectivity and pure-injective modules over Prüfer rings. *J. Algebra*, 110:380–406, (1987).
- [4] I. Kaplansky. *Infinite Abelian Groups*. the University of Michigan Press, Ann Arbor, (1969).
- [5] D. Simson. Dualities and pure semisimple rings. In *Abelian groups, module theory and topology*, volume 201 of *Lecture Notes in Pure and Appl. Math.*, pages 381–388. Marcel Dekker, (1998).
- [6] B. Zimmermann-Huisgen and W. Zimmermann. Algebraically compact rings and modules. *Math. Z.*, 161:81–93, (1978).

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